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ON THEOREMS OF DYNAMICS

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One parameter families of transformations of the configurational space are considered, and possible displacements compatible with the constraints imposed on the system, introduced. The D'Alembert—Lagrange principle is applied to these displacements to obtain assertions generalizing the fundamental theorems of dynamics and extending the Noether's theorem to a system with nonholonomic constraints. The assertions proved in the paper are used to solve a problem of motion of a sharp-edged homogeneous disc on a horizontal ice surface.

1. Generalization of the fundamental theorems of dynamics. Let us consider a mechanical system of n material points of masses m_i the Cartesian coordinates of which are x_i, y_i, z_i. We assume that the system is under linear constraints which are, in general, nonintegrable. Then the possible displacements of the system satisfy the relations

$$\sum_{i=1}^{n}(a_{ij}\delta x_i + b_{ij}\delta y_i + c_{ij}\delta z_i) = 0, \qquad j = 1, \ldots, \quad m < 3n \tag{1.1}$$

where the coefficients are functions of the coordinates and time. The points m_i are acted upon by the active forces $\mathbf{F}_i$ whose projections on the coordinate axes are X_i, Y_i, Z_i. The actual motions can be found from the D'Alembert — Lagrange principle

$$\sum_{i=1}^{n}\left[\left(m_i\frac{d^2x_i}{dt^2} - X_i\right)\delta x_i + \left(m_i\frac{d^2y_i}{dt^2} - Y_i\right)\delta y_i + \left(m_i\frac{d^2z_i}{dt^2} - Z_i\right)\delta z_i\right] = 0 \tag{1.2}$$

Let us consider the time - and parameter α -dependent family of reversible transformations of a $3n$ -dimensional configurational space

$$\mathbf{r}_i = \mathbf{r}_i(x_1', y_1', z_1', \ldots, x_n', y_n', z_n', t, \alpha) \tag{1.3}$$

$$\mathbf{r}_i = \{x_i, y_i, z_i\}$$

The velocities of the points of the system are transformed in accordance with the accepted rule

$$\frac{d\mathbf{r}_i}{dt} = \frac{\partial\mathbf{r}_i}{\partial t} + \sum_{j=1}^{n}\left(\frac{\partial\mathbf{r}_i}{\partial x_j'}\frac{dx_j'}{dt} + \frac{\partial\mathbf{r}_i}{\partial y_j'}\frac{dy_j'}{dt} + \frac{\partial\mathbf{r}_i}{\partial z_j'}\frac{dz_j'}{dt}\right) \tag{1.4}$$

We define

$$\delta\mathbf{r}_i = \frac{\partial\mathbf{r}_i}{\partial\alpha}\delta\alpha \tag{1.5}$$

as the possible displacements of the system, connected with the family of transformations (1.3). We shall say that the family of transformations (1.3) is compatible with the constraints (1.1) if the possible transformations (1.5) satisfy the constraint equations (1.1).

Lemma 1. We have the following relation:

$$\sum_{i=1}^{n}\left(m_i \frac{d^2x_i}{dt^2}\frac{\partial x_i}{\partial \alpha} + m_i \frac{d^2y_i}{dt^2}\frac{\partial y_i}{\partial \alpha} + m_i \frac{d^2z_i}{dt^2}\frac{\partial z_i}{\partial \alpha}\right) = \frac{dS}{dt} - \frac{\partial T}{\partial \alpha} \tag{1.6}$$

$$S = \sum_{i=1}^{n}\left(\frac{\partial T}{\partial x_i^{\cdot}}\frac{\partial x_i}{\partial \alpha} + \frac{\partial T}{\partial y_i^{\cdot}}\frac{\partial y_i}{\partial \alpha} + \frac{\partial T}{\partial z_i^{\cdot}}\frac{\partial z_i}{\partial \alpha}\right)$$

$$T = \sum_{i=1}^{n}\frac{m_i}{2}\left[(x_i^{\cdot})^2 + (y_i^{\cdot})^2 + (z_i^{\cdot})^2\right]$$

(where T is the kinetic energy). The validity of this identity follows from the permutation relations:

$$\frac{d}{dt}\frac{\partial \mathbf{r}_i}{\partial \alpha} = \frac{\partial}{\partial \alpha}\frac{d\mathbf{r}_i}{dt}$$

Lemma 2. If the family (1.3) is compatible with the constraints placed on the system, then

$$\frac{dS}{dt} - \frac{\partial T}{\partial \alpha} = \sum, \quad \sum = \sum_{i=1}^{n}\left(X_i \frac{\partial x_i}{\partial \alpha} + Y_i \frac{\partial y_i}{\partial \alpha} + Z_i \frac{\partial z_i}{\partial \alpha}\right) \tag{1.7}$$

We prove this relation by substituting the possible displacements (1.5) into the D'Alembert—Lagrange equations and applying the formula (1.6).

The function $f(t, x_1, y_1, z_1, \ldots, x_1^{\cdot}, y_1^{\cdot}, z_1^{\cdot}, \ldots)$ is invariant under the transformations (1.3) provided that the function g obtained from f by changing the coordinates and velocities of the points in accordance with (1.3) and (1.4), is independent of α.

Theorem 1. If the kinetic energy is invariant under the property of transformations (1.3) compatible with the constraints, then

$$dS/dt = \Sigma \tag{1.8}$$

Since the kinetic energy is invariant under the family of transformations (1.3), $\partial T / \partial \alpha = 0$ and (1.8) follows from (1.7).

The kinetic energy is invariant under the displacements along a fixed direction and rotations about a fixed axis. Consequently, in these cases Theorem 1 coincides with the classical theorems on the change of momentum and moment of momentum of the system [1].

2. **The case of potential forces.** We assume that the external forces $\mathbf{F}_i$ acting on the system admit the force function $V(t, x_1, y_1, z_1, \ldots x_n, y_n, z_n)$.

Theorem 2. If the function $L = T + V$ is invariant under the family of transformations (1.3) together with the constraints, the equations of motion have the first integral

$$S = \text{const}$$

This assertion follows from the formulas (1.7), and we make use of the formula

$$\Sigma = \partial V / \partial \alpha$$

and the invariance of the function $L = T + V$: $\partial L / \partial \alpha = 0$.

When the constraints are nonholonomic, Theorem 2 becomes identical with the Noether's theorem [2]. We stress that the family of transformations (1.3) need not be a group.

3. Generalization of the theorems on determination of momentum and moment of momentum. In this section we show when the kinetic energy is invariant under the displacements along the straight line l defined by the direction cosines a, b, c, which can change its direction with time. Obviously, in this case the transformation formulas are

$$\mathbf{r}_i = \mathbf{r}_i' + \alpha \mathbf{l}, \quad \mathbf{l} = (a, b, c)$$

It can be verified that

$$\frac{\partial T}{\partial \alpha} = \frac{da}{dt} \sum_{i=1}^{n} m_i \frac{dx_i}{dt} + \frac{db}{dt} \sum_{i=1}^{n} m_i \frac{dy_i}{dt} + \frac{dc}{dt} \sum_{i=1}^{n} m_i \frac{dz_i}{dt}$$

In this case the condition of invariance of T relative to the family of displacements are written in the form $(\mathbf{P}, d\mathbf{l} / dt) = 0$ where $\mathbf{P}$ is the vector of the momentum of the system. If the above relation holds and the constraints allow a translational displacement of the system along the l-axis as a single rigid body, then according to Theorem 1 we have

$$\frac{d}{dt}(\mathbf{P}, \mathbf{l}) = \left(\sum_{i=1}^{n} \mathbf{F}_i, \mathbf{l}\right) \tag{3.1}$$

We shall also show when the kinetic energy is invariant relative to rotation about the l-axis which is, in general, movable. Once again we denote its direction cosines by a, b, c and allow l to pass through the coordinate point x_0, y_0, z_0. The quantities a, b, c, x_0, y_0 and z_0 are known functions of time.

Let α denote the angle of rotation. It can be verified that

$$\frac{\partial T}{\partial \alpha} = \sum_{i=1}^{n} m_i \frac{dx_i}{dt} \frac{d}{dt} [b(z_i - z_0) - c(y_i - y_0)] +$$
$$\sum_{i=1}^{n} m_i \frac{dy_i}{dt} \frac{d}{dt} [c(x_i - x_0) a - (z_i - z_0)] +$$
$$\sum_{i=1}^{n} m_i \frac{dz_i}{dt} \frac{d}{dt} [a(y_i - y_0) - b(x_i - x_0)]$$

After the transformations, we can write the condition of invariance of the functions relative to the family of rotations in the form

$$\left(\mathbf{P}, \frac{d}{dt}[\mathbf{r}_0, \mathbf{l}]\right) + \left(\mathbf{K}, \frac{d\mathbf{l}}{dt}\right) = 0, \quad \mathbf{r}_0 = (x_0, y_0, z_0) \tag{3.2}$$

where $\mathbf{K}$ is the vector of angular momentum of the system relative to the coordinate origin.

If the constraints allow the rotation of the system about the l -axis as a single rigid body and the relation (3.2) holds, then by Theorem 1 we have

$$d/dt\,(\mathbf{K}', \mathbf{l}) = (\mathbf{M}', \mathbf{l}) \tag{3.3}$$

where $\mathbf{K}'$ and $\mathbf{M}'$ denote, respectively, the angular momentum and the total moment of forces about the point (x_0, y_0, z_0). In other words, if condition (3.2) is satisfied, the theorem on the change of moment of momentum holds for the moving l -axis. If, in particular, the l -axis does not change its direction in space, i.e. $da/dt = db/dt = dc/dt = 0$, the assertion becomes identical to the generalization of the theorem of areas [3, 4].

When the l -axis passes through the center of gravity of the system, the condition (3.2) can be simplified to

$$\left(\mathbf{K}', \frac{d\mathbf{l}}{dt}\right) = 0 \tag{3.4}$$

4. Example from the dynamics of nonholonomic systems. We illustrate the application of the assertions proved above by considereing the problem of motion of a circular disc with a sharp edge along a smooth horizontal ice surface. This is equivalent to imposing a nonholonomic constraint on the system; the constraint being that the velocity of the point of contact of the disc is parallel to its horizontal diameter. The disc is assumed dynamically symmetric, and its center of gravity coincides with its geometrical center.

We introduce the Koenig $Ox_1y_1z_1$-axes with the Oz_1-axis vertical. In another $Oxyz$ moving coordinate system the Oz -axis is perpendicular to the disc plane, the Ox -axis is horizontal and the Oy -axis passes through the point of contact H. We denote by M a point on the circumference of the disc. Let m be the mass of the disc, a its radius, and A, C its moments of inertia about the Ox -, Oy - and Oz -axes. We also denote the projections of the angular velocity of the disc ω on the $Oxyz$ -axes by p, q, r, and projections of the velocity of the mass center on the same axes by u, v, w, respectively. Projecting the velocity $V_H = V_0 + [\omega, OH]$ on the axes of a trihedron and using the fact that V_H is parallel to the Ox -axis we obtain

$$v = 0, \quad w - ap = 0 \tag{4.1}$$

We shall prove that $r = \text{const.}$, taking the Oz -axis as the moving l -axis. The constraints allow the disc to rotate about this axis.

Let us show that the condition (3.4) holds. Indeed, the projections of $\mathbf{K}'$ on the $Oxyz$ axes are Ap, Aq, Cr, and those of the vector $d\mathbf{l}/dt$ are $q, -p, 0$; hence they are orthogonal. Since the forces of gravity has zero moment about the Oz -axis, we find from (3.3) that $d(Cr)/dt = 0$, i.e. $r = r_0 = \text{const.}$

The constraints allow the disc to rotate about the vertical Oz_1-axis. Since O is the center of gravity, then according to the formula (3.3) (Koenig's theorem) the projection of the moment of momentum relative to the point O on the vertical, is constant

$$Aq \sin\theta + Cr \cos\theta = c_1 \quad \text{or} \tag{4.2}$$

$$q(\theta) = \frac{c_1}{A \sin\theta} - \frac{Cr_0}{A} \operatorname{ctg}\theta$$

The disc can undergo a translational motion along the moving Ox -axis. The kinetic energy of the disc is not invariant under these displacements, nevertheless Lemma 2 can be utilized to yield the equation $m\,du/dt - mwq = 0$ or, together with (4.1) $du/dt - aqp = 0$. Since $p = d\theta/dt$, taking into account (4.2) we obtain

$$du = a\left(\frac{c_1}{A\sin\theta} - \frac{Cr_0}{A}\operatorname{ctg}\theta\right)d\theta \tag{4.3}$$

and hence

$$u(\theta) = c_2 + \frac{ac_1}{A}\ln\operatorname{tg}\frac{\theta}{2} - \frac{aCr_0}{A}\ln\sin\theta$$

The total energy of the disc is conserved

$$\tfrac{1}{2}m(u^2 + v^2 + w^2) + \tfrac{1}{2}(Ap^2 + Aq^2 + Cr^2) + mga\sin\theta = h$$

Taking into account the relations (4.1) - (4.3), we can write the above equation in the form

$$\frac{1}{2}(A + ma^2)\left(\frac{d\theta}{dt}\right)^2 = h - mga\sin\theta - \frac{m}{2}u^2(\theta) - \frac{A}{2}q^2(\theta) - \frac{Cr_0^2}{2} \tag{4.4}$$

and this yields the angle θ using the method of quadratures.

If $c_1 \neq Cr_0$, then the right-hand side of (4.4) tends to $-\infty$ as $\theta \to 0, \pi$. Consequently in this case we have $0 < \theta < \pi$ and $\theta(t)$ is a function of time with a period τ. It follows, in particular, that the disc can never fall onto the surface. The disc can fall when $c_1 = Cr_0$, but only when it is not placed vertically and is released at zero initial velocity.

Let us assume that $c_1 \neq Cr_0$. Then p, q, r, u, v, w are τ -periodic functions of time. To complete the qualitative pattern of motion we must explain how the angles φ between OH and OM and ψ between Ox and Ox_1 vary with time, and to find the law of motion of the point of contact along the surface.

From the kinematic relations

$$q = \frac{d\psi}{dt}\sin\theta, \quad r = \frac{d\varphi}{dt} + \frac{d\psi}{dt}\cos\theta$$

we find that

$$\psi = \lambda_1 t + f_1(t), \quad \varphi = \lambda_2 t + f_2(t) \tag{4.5}$$

where λ_1 and λ_2 are constants depending on the initial conditions and f_1, f_2 are τ -periodic functions of time.

Let ξ and η be Cartesian coordinates of the point of contact H on the surface, and let the ξ, η -axes be parallel to the Ox_1, Oy_1 -axes. It can be shown that

$$\frac{d\xi}{dt} = u\cos\psi + w\sin\theta\sin\psi + \frac{d}{dt}(a\cos\theta\sin\psi) = \left(u + a\cos\theta\frac{d\psi}{dt}\right)\cos\psi$$

$$\frac{d\eta}{dt} = u\sin\psi - w\sin\theta\cos\psi - \frac{d}{dt}(a\cos\theta\cos\psi) = \left(u + a\cos\theta\frac{d\psi}{dt}\right)\sin\psi$$

The function $u + a\cos\theta\, d\psi/dt$ is τ -periodic in t. Let us denote it by $g(t)$. Then, taking (4.5) into account we obtain

$$d\zeta / dt = g(t) \exp [i (\lambda_1 t + f_1 (t))], \qquad \zeta = \xi + i\eta$$

The function $g(t) \exp [if_1 (t)]$ is τ -periodic. Let us expand it into a converging Fourier series

$$\sum_{-\infty}^{\infty} a_n \exp\left(i \frac{2\pi n}{\tau} t\right)$$

Then we have

$$\zeta = c + \sum_{-\infty}^{\infty} \frac{a_n}{i(2\pi n / \tau + \lambda_1)} \exp\left(i \frac{2\pi n t}{\tau}\right) \exp (i\lambda_1 t)$$

where $c = c_1 + ic_2$ is a constant. If $2\pi n / \tau + \lambda_1 \neq 0$ for integral n, then

$$G(t) = \sum_{-\infty}^{\infty} \frac{a_n}{i(2\pi n / \tau + \lambda_1)} \exp\left(i \frac{2\pi n}{\tau} t\right)$$

is an analytic τ -periodic function. In this case $\zeta = G(t) \exp (i\lambda_1 t) + c$. Let us introduce a frame of reference rotating with the angular velocity λ_1 about the point c, and the point $\zeta(t)$ will undergo a periodic motion along a closed analytic curve $\zeta = G(t)$. In the stationary (ξ, η) -plane the point of contact will perform a complex periodic motion along a closed analytic curve, rotating in a manner of a rigid body with constant angular velocity about a fixed point.

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